6th Aug, 20..

B.Sc. Part III 5th Paper

Infinite Products (contd.)

Q: Prove that the product

$\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\left(1-\frac{1}{5}\right)\cdots$ converges to 1.

$\boxed{OR}$

$\prod_{1}^{\infty}\left(1+\frac{1}{2n}\right)\left(1-\frac{1}{2n+1}\right)$ is convergent.

Solution: The n^{th} term of $\left(1+\frac{1}{2}\right), \left(1+\frac{1}{4}\right)\cdots = 1+\frac{1}{2n}$

and n^{th} term of $\left(1-\frac{1}{3}\right), \left(1-\frac{1}{5}\right)\cdots = 1-\frac{1}{2n+1}$

Let $P_{2n} = \left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\left(1-\frac{1}{5}\right)\cdots\left(1+\frac{1}{2n}\right)\left(1-\frac{1}{2n+1}\right)$

$\Rightarrow P_{2n} = \frac{3}{2}\cdot\frac{2}{3}\cdot\frac{5}{4}\cdot\frac{4}{5}\cdots\frac{2n+1}{2n}\cdot\frac{2n}{2n+1} = 1$

$P_{2n+1} = P_{2n}\cdot\left(1+\frac{1}{2n+2}\right) = P_{2n}\cdot\frac{2n+3}{2n+2}$

Take Limit $n\to\infty$

$\Rightarrow \underset{n\to\infty}{Lt} P_{2n+1} = \lim_{n\to\infty}\left[P_{2n}\cdot\left(\frac{2n+3}{2n+2}\right)\right]$

$\Rightarrow \underset{n\to\infty}{Lt} P_{2n+1} = \left(\underset{n\to\infty}{Lt} P_{2n}\right)\left(\underset{n\to\infty}{Lt}\frac{2n+3}{2n+2}\right)$

$= 1\cdot\underset{n\to\infty}{Lt}\left(\frac{2+\frac{3}{n}}{2+\frac{2}{n}}\right) = 1\cdot\frac{2}{2} = 1$

$\therefore \underset{n\to\infty}{Lt} P_{2n+1} = 1$. Hence, the given infinite product converges to 1.

Q. Prove that the product

$$\left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1+\frac{1}{5}\right)\cdots$$ converges to $\frac{1}{2}$.

Soln

Let a_n = n^{th} factor of the product

$= \left(1-\frac{1}{1+n}\right)$ if n is odd i.e. $\frac{n}{1+n}$ when n is odd

and $a_n = 1+\frac{1}{n+1}$ if n is even

$= \frac{n+2}{n+1}$ if n is even.

$\therefore P_n$ = product of n factors

$= \left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1+\frac{1}{5}\right)\cdots\left(1-\frac{1}{1+n}\right)$, n is odd

$= \frac{1}{2}\cdot\cancel{\frac{4}{3}}\cdot\cancel{\frac{3}{4}}\cdot\cancel{\frac{5}{4}}\cdots\cancel{\frac{n+1}{n}}\cdot\cancel{\frac{n}{n+1}} = \frac{1}{2}$

$\Rightarrow \underset{n\to\infty}{Lt} P_n = \frac{1}{2}$

and $P_n = \left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1+\frac{1}{5}\right)\cdots\left(1+\frac{1}{n+1}\right)$, n is even

$= \frac{1}{2}\cdot\cancel{\frac{4}{3}}\cdot\cancel{\frac{3}{4}}\cdot\cancel{\frac{6}{5}}\cdots\cancel{\frac{n}{n-1}}\cdot\cancel{\frac{n-1}{n}}\cdot\frac{n+2}{n+1}$

$\Rightarrow P_n = \frac{1}{2}\cdot\frac{n+2}{n+1}$

$\Rightarrow \underset{n\to\infty}{Lt} P_n = \frac{1}{2}\cdot\underset{n\to\infty}{Lt}\frac{n+2}{n+1} = \frac{1}{2}\cdot\underset{n\to\infty}{Lt}\frac{1+\frac{2}{n}}{1+\frac{1}{n}}$

$\Rightarrow \underset{n\to\infty}{Lt} P_n = \frac{1}{2}$. Hence $\underset{n\to\infty}{Lt} P_n = \frac{1}{2}$, n = even or odd.

Hence the given product converges to $\frac{1}{2}$.